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Short communication

The optimal portfolio model based on multivariate t distribution with linear weighted sum method

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This paper proposed the optimal portfolio model maximizing returns and minimizing the risk expressed as CvaR under the assumption that the portfolio yield is subject to the multivariate t distribution. With linear weighted sum method, we solved the multi-objectives model, and compared the model results to the case under the assumption of normal distribution return, based on the portfolio VAR through empirical research. It is showed that our max return equals to and risk is higher than M-V model. It shows that CVaR predicts the potential risk of the portfolio, which is helpful for investor's cautious investment.

Keywords: Multivariate t distribution; The optimal portfolio; CVAR; Multi- objectives programming; Linear weighted sum method

INTRODUCTION

Portfolio optimization has come a long way from Markowitz (1952) seminal work which introduces return/variance risk management framework. One line of work has focused on the assumption that portfolio return subject to normal distribution, but there is substantial empirical evidence which shows that financial returns exhibit fat-tails and excess kurtosis after accounting for the clustering of volatility and autocorrelation. Using different approaches to the problem and different sets of data, these studies consistently find high kurtosis and heavy tails, different models have been suggested to explain these empirical facts. Mandelbrot (1963) and Fama (1965) proposed the stable Paretian distribution which was later incorporated as a building block in GARCH-type processes, see for example Mittnik et al., (2002) and Mittnik and Paolella(2003). The Student's t model is a common portfolio returns assumption. There are two reasons for choosing stable distributions. Firstly, because their tails are thicker than the Student's t tails, we can see how the relative importance of the tail thickness changes. Second, stable distributions are a very heavy tailed model. This implies that the relative importance of the tail thickness will be weaker for any other distributional model with tails decaying faster than the tails of stable laws. Another line of work has focused on developing more realistic models of changes in risk factors. As a supplement (or alternative) to VaR, another percentile risk measure which is called Conditional Value-at-Risk(CVaR), which is defined as the conditional expected loss under the condition that it exceeds VaR, see Rockafellar and Uryasev (2000). It has been shown (Pflug, 2000) that CVaR is a coherent risk measure that has many attractive properties including convexity, e.g., see Ogryczak and Ruszczynski (2002) for an overview of CVaR. Although CVaR means the conditional mean of the loss of Var, which is better to satisfy the additive need, and showing monotony, if we choose an appropriate distribution of return combining with CVaR, the model will be closer to reality. Bollerslev (1987) described the foreign exchange return with t-distribution firstly. But they did not consider the skewed distribution, so income distribution also caused changes in portfolio risk change with the characterization. Hansen (1994) proposed skewed-t-distribution firstly, and considered both capital gains and fat-tail of the skewed nature of consideration. In recent research, some studies extended the single-variable distribution to multi-variate distribution, such as introducing copula function.

This paper contributes to both lines of investigation by developing methods for calculating portfolio maximizing returns and minimizing the risk expressed as CvaR under the assumption that the portfolio yield subject to multivariate t distribution . Different from other literatures, we considered the continuous case, calculating CVaR

formula by multivariate t distribution density function.
This paper is organized as follows. In Section 2, the notions of multivariate t distribution and CVaR are introduced. In Section 3, the optimal portfolio model is given. In Section 4, an empirical study is performed and compared the result to the case of Mean- variance model and conclusion in section 5.

Multivariate t distribution

Suppose that $X=(T_1(\nu),T_2(\nu)\cdots T_n(\nu))$

n =dimension student; t =distribution，with ν = degrees of freedom, its probability density function is [5]：

$$p(y)=\frac{\Gamma\left(\frac{\nu+n}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{|V|}(\pi\nu)^{\frac{n}{2}}}\cdot\left(1+\frac{1}{\nu}(y-\mu)^T V^{-1}(y-\mu)\right)^{-\frac{\nu+n}{2}} \tag{1}$$

$$=|V|^{-\frac{1}{2}} f\left((y-\mu)^T\frac{V^{-1}}{\nu}(y-\mu)\right)$$

$$f(u)=\frac{\Gamma\left(\frac{\nu+n}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)(\pi\nu)^{\frac{n}{2}}}\cdot\left(1+\frac{u}{\nu}\right)^{-\frac{\nu+n}{2}}$$

That is：

$$A=\begin{pmatrix}1&0&\cdots&0\\0&1&\cdots&0\\0&0&\cdots&0\\0&0&\cdots&1\end{pmatrix}$$

let $\mu=(\mu_1,\mu_2\cdots\mu_n)$,

$Y=AX+\mu$, $V=\frac{\nu}{\nu-2}AA^T$,

The portfolio returns is $R=w^TY$，and $E(R)=w^T\mu$ where $\mu^T=(\mu_1,\mu_2\cdots\mu_n)$ are yield vector of n assets，the investment share $w^T=(w_1,w_2\cdots w_n)$ with constraint $\sum_{i=1}^{n} w_i=1, 0\le w_i\le 1$.

Conditional Value-at-Risk

For each w, the loss $f(w,x)$ is a random variable having a distribution induced by that of x. The underlying probability distribution of x will be assumed for convenience to have density, which we denote by: p(x) as (1).

For a portfolio w，the loss is defined $f(w,x)=-w^Tx$，given a believe degree $\beta(0<\beta<1)$，$VaR_\beta(w)$, $CVaR_\beta(w)$ are defined as[7]：

$$VaR_\beta(w)=\min\left\{\alpha\left|\int_{f(w,x)\le\alpha}p(x)dx\ge\beta\right.\right\}$$

$$CVaR_\beta(w)=(1-\beta)^{-1}\int_{f(w,x)\ge VaR_\beta(x)}f(w,x)\,p(x)dx$$

Value-at-Risk under multi-continuous-t case

Lemma：under the suppose of multivariate t distribution，with believe degree $\beta(0<\beta<1)$：

$$VaR(w)=-w^T\mu+S\left|w^TVw\right|^{\frac{1}{2}}$$

$$CVaR(w)=-w^T\mu+H\left|w^TVw\right|^{\frac{1}{2}}$$

where：

$$H=\frac{\Gamma\left(\frac{\nu-1}{2}\right)}{2(1-\alpha)\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi}}\nu^{\frac{\nu}{2}}\left(\left(\frac{w^T\mu+VaR}{\left|w^TVw\right|^{\frac{1}{2}}}\right)^2+\nu\right)^{-\frac{\nu-1}{2}}$$

and S is solved by the equation:

$$\alpha=\frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)}\int_S^{+\infty}\int_{z^2}^{+\infty}\left(u-z^2\right)^{\frac{n-3}{2}}f(u)\,du\,dz$$

Proof: Changing variables to $y=(x-\mu)A^{-1}$, $dy=|A|dx$, where $\Sigma=A^TA$ is a Cholesky decomposition of A ,this becomes

$$\beta=\int_{\{\delta A\cdot y\le-\delta\cdot\mu-VaR_\beta\}}f\left(|y|^2\right)dy$$

Let R be a rotation which sends δA to $(|\delta A|,0\cdots0)$ Changing variables once more to $y=zR$, we obtain the equation

$$\beta=\int_{\{|\delta A|z_1\le-\delta\cdot\mu-VaR_\beta\}}f\left(|z|^2\right)dz$$

If we write that $|z|^2=z_1{}^2+\left|z_1'\right|^2$ with $z'\in R^{n-1}$ then we have shown that :

$$\beta = \Pr ob\{\delta \cdot X < -VaR_\beta\} = \int_{R^{n-1}} \left[\int_{+\infty}^{\frac{-\delta\mu - VaR_\beta}{|\delta A|}} f\left(z_1^2 + |z'|\right)^2 dz_1 \right] dz'$$

Next, by using spherical variables $z' = r\xi$ with $\xi \in S_{n-2}, dz' = r^{n-2} d\sigma(\xi) dr$, we see that we have to solve for VaR_β in the equation

$$\beta = |S_{n-2}| \int_0^{+\infty} r^{n-2} \left[\int_{-\infty}^{\frac{-\delta\mu^T - VaR_\beta}{|\delta A|}} f\left(z_1^2 + r^2\right) dz_1 \right] dr$$

$|S_{n-2}|$ being the surface measure of the unit-sphere in R^{n-1}:

$$|S_{n-2}| = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)}$$

We now introduce the function

$$G(S) = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)} \int_{-\infty}^{-S} \left[\int_0^{+\infty} r^{n-2} f\left(z_1^2 + r^2\right) dr \right] dz_1$$

$$= \frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)} \int_s^{-\infty} \left[\int_{z_1^2}^{+\infty} \left(u - z_1^2\right)^{\frac{n-3}{2}} r^{n-2} f(u)\, du \right] dz_1$$

where for the second line we changed:

variables $u = r^2 + z_1^2$ replaced z_1 by $-z_1$. We then have proved the lemma result.

The optimal portfolio under Mean-CVaR model

For investors, the aim is to seek to maximize returns while controlling risk as minimal risk. Suppose that the returns of portfolio follow multivariate t distribution. We propose the optimal portfolio under Mean-CVaR model:

$$(P1)\begin{cases} \max \ \ E(R) = w^T \mu & (1) \\ \min \ \ CVaR(w) & (2) \\ s.t \ \sum_{i=1}^{n} w_i = 1 & (3) \\ 0 \le w_i \le 1 & (4) \end{cases}$$

Model $(P1)$ is a multi-objective optimization problem, it is solved by the linear weighted sum method, the steps are:

(1) Construct a single objective function $\max(u(w)) = \max(\alpha_1 E(R) - \alpha_2 CVaR)$, investors can choose different weights α_1, α_2 according to their preference. For example, if investors are more concerned about income compared to risk, they can make α_1 higher than α_2. Similarly, if investors are more concerned about risk compared to income, they can make α_2 α_1 higher than α_1. In this paper, we choose $\alpha_1 = 0.5,\ \alpha_2 = 0.5$.

(2) Under the constrains $0 \le w_1 \le 1; \quad 0 \le w_2 \le 1$ $w_1 + w_2 = 1$, we solved the single-objective programming problem $\max_w u(w)$.

Empirical study

Mean-CVaR model under multivariate t distribution

Through the history of the stock closing price of each stock we can calculate skewness and kurtosis. In this paper, we need to collect a peak degree of stock, for example, we choose two stocks（MS and Google）,date begins2008.1.1 to 2011.8.26, with 922 closed days, calculate each day yield:

$$y_{i,j} = \frac{p_{i,j} - p_{i,j-1}}{p_{i,j-1}},$$

where $p_{i,j}$ is previous day's closing price and $p_{i,j-1}$ is the day after.And we also can get the average yield of MS is $\mu_1 = 0.0001789$, kurtosis is 70.288; the average yield of Google is $\mu_2 = 0.0000044$, kurtosis is 9.693, compare to normal return's kurtosis 3,the two stocks returns's is larger ,that is, the return is heavy tails.To solve model $(P1)$, we get the optimal portfolio is $w = [1,0]$, in which case, $\max E(w) = 0.0001789$, $\min CVaR(w) = 0.020849$

Mean-VaR model under normal distribution

We can also solve Markowitz model, and get the optimal portfolio is $w = [0.99, 001]$, in which case:

$\max E(w) = 0.0001789$,

$\min VaR(w) = 0.003$

It is showed that our max return is equal to and risk is higher than M-V model. So the CVaR predicts the potential risk of the portfolio, which helps investor cautious investment.

CONCLUSION

In this paper we first considered the thick tail of the portfolio income, which is illustrated with the peak degree，and in the portfolio model maximizing portfolio returns, while the minimizing risk characterized as CVaR，which is more meaningful than a single objective for investors. Although the empirical study selected only two stocks, but the model can be more easily extended to the case of many stocks. In addition, we propose linear weighted sum method to solve the model, it is more efficient, and it is more flexible for investors to choose different weights according to their investment preferences and risk.

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